UNIT-01

11/07/2025 Friday

UNIFORM, BERNOULLI AND BINOMIAL DISTRIBUTIONS

*Uniform distribution :-

Introduction :

This is the simplest possible probability distribution it describes a variable than can take one of several explicit discrete values with equal probabilities of taking any particular value. when the discrete random variable can take the n values, say $x_1, x_2 \ldots x_n$ with equal probabilites $1/n$, it is called an rectangular distribution or uniform distribution. This distribution can be represented by the formula

$$\boxed{P(X = x_r) = 1/n \quad , r = 1, 2, \ldots n}$$

This distribution is characterized by a single parameter n, which is the no. of different values that can be taken by a random variable. The above probability distribution has to satisfy the following conditions.

1. Each probability, being equal to $1/n$, lies between 0 and 1.
2. The sum of probabilities ~~equal to~~

$$= \frac{1}{n} + \frac{1}{n} + \cdots n \text{ terms}$$

$$= \frac{n}{n} = 1$$

Examples:

1. If a fair die if thrown then the possible values of X are 1, 2, 3, 4, 5, 6 and each time the probability is $\frac{1}{6}$ $p(x) = 1/6$

$$X = 1, 2, 3, 4, 5, 6$$

* Expected value, Variable and mode:

$$\text{Mean} = \mu_1' = \sum_x x\, p(x)$$

$$= \sum_{x=1}^{n} x \cdot \frac{1}{n}$$

$$= \frac{1}{n}[1 + 2 + \cdots n]$$

$$= \frac{n+1}{2}$$

07/07/2025 Monday

$$\mu_2' = E[X^2] = \sum_{x=1}^{n} x^2 p(x)$$

$$= \sum x^2 \frac{1}{n}$$

$$= \frac{1}{n}[1^2 + 2^2 + \cdots n^2]$$

$$= \frac{1}{n}\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{(n+1)(2n+1)}{6}$$

$$\text{Varience} = E[x^2] - [E(x)]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \left[\frac{n+1}{2}\right]^2$$

$$= \frac{2n^2+3n+1}{6} - \frac{n^2+2n+1}{4}$$

$$= \frac{4n^2+\cancel{6n}+2-3n^2-\cancel{6n}-3}{12}$$

$$= \frac{n^2-1}{12}$$

Imp

*Bernouli Distribution :-

Introduction : A random experiment is one that resulting either of two possible outcomes is called Bernouli Experiment.

There are many examples of such experiments. Some of the examples are tossing a coin (results a Head & a tail), performance of a student in an exam (results pass & fail).

The probability of success say P, remains the same from trail to trail when Bernouli experiment is performed several independent times. the probability distribution of an Bernouli trail is depends only on the single parameter p

Definition: let x be a random variable defined

On Bernouli trail. The outcomes can be defined,
X (Success) = 1 and X (Failure) = 0

The probability mass function of X can be written as

$$p(X=x) = \begin{cases} p^x (1-p)^{1-x}, & x = 0,1 \\ 0, & \text{otherwise} \end{cases} \quad (1-p) = q$$

Mean and variance of Bernouli Distribution:

X	1	0
p(x)	p	q

$$\text{Mean} = E(X) = \Sigma x\, p(x) = \Sigma x p^x \Sigma x q^{1-x}$$

$$= 1 \cdot p + 0 \cdot q = p$$

$$E(X^2) = 1^2 \cdot p + 0^2 \cdot q = p^2$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$= p - p^2 = p(1-p) = pq$$

Imp

★ Binomial Distribution:-

Introduction: Binomial Distribution was discovered by "James Bernouli's". this is the most widely used discrete probability Distribution. If an random experiment is repeated 'n' times with two mutual Exclusive outcomes occurence (Success) and non-occurence (Failure), in each trail.

The total no. of success is a random variable with value $0, 1, 2, \dots n$. In this situation if we treat every trail has Bernouli, then and independent Bernouli trail with the probability of success is constant in each trail is called ~~Bernoulli~~ Binomial experiment.

Imp

physical conditions of ~~Bernoulli~~ Binomial Distribution:

1. No. of trails must be infinite.
2. Each trail result in two mutual Exclusive outcomes (Success and failure).
3. The trails are independent.
4. The probability of success p is constant in each trail.

Imp

Definition: A discrete random variable X assumes the value $0, 1, 2, \dots n$ with probability mass function

$$P(X=x) = \begin{cases} {}^{n}C_{x}\, p^{x} q^{n-x}, & x = 0, 1, 2, \dots n \\ 0; & \text{otherwise} \end{cases}$$

is called ~~Bernoulli~~ Binomial distribution with parameters n, p. It ~~is~~ Symbiolically it is denoted as $X \sim B(n, p)$

Derivation: Binomial distribution can be derived in the following way let us consider 'n' indepent Bernouli trails with constant probability of success p and probability of failure $Q = 1-p$ in each trail.

The probability of getting x success and remaining $n-x$ failures is $p \cdot p \cdot q \cdot q \cdot p \cdot p \cdot p \cdot p \cdot q \cdot q \cdot q \cdot q$

$$\underbrace{(p\,p \cdots p)}_{x \text{ success}}\ \underbrace{(q\,q\,q \cdots q)}_{n-x \text{ failure}}$$

x success in n trails can be obtained in ^nC_x ways or $\binom{n}{x}$. Hence probability of getting exactly x success in n traile

$$P(X=x) = \binom{n}{x} p^x q^{n-x}$$

Note: $\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (p+q)^n$, $p+q=1$

Imp

Moments of Binomial distribution:

We know that the r^{th} moment about origin is denoted by μ_r' and is defined as

$$\mu_r' = E(x^r)$$

$$\mu_r' = \sum x^r p(x)$$

if $r=1$

$$^nC_x = \frac{n!}{(n-x)!\,x!}$$

$$\mu_1' = \sum_{x=0}^{n} x\,p(x)$$

$$= \sum_{x=0}^{n} x\ ^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x \cdot \frac{n}{x}\ ^{n-1}C_{x-1}\, p^{x-1}\, p\, q^{(n-1)-(x-1)}$$

$$= np \sum_{x=0}^{n} {}^{n-1}c_{x-1}\, p^{x-1} q^{(n-1)(x-1)}$$

$$= np\,(p+q)^{n-1} \qquad \left[\sum \binom{n}{x} p^{x} q^{n-x} = (p+q)^{n} \right]$$

$$\boxed{\mu_1' = np} \text{ - mean}$$

If $r = 2$

$$\mu_2' = \sum_{x=0}^{n} x^2 p(x)$$

$$= \sum_{x=0}^{n} x^2 \cdot \binom{n}{x} p^{x} \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} (x(x-1)+x) \binom{n}{x} p^{x} \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \binom{n}{x} p^{x} q^{n-x} + \sum_{x=0}^{n} x\, {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n(n-1)}{x(x-1)}\, {}^{n-2}C_{x-2}\, p^{x-2} p^{2} q^{(n-2)-(x-2)} + np \quad \text{(from ①)}$$

$$= \sum_{x=0}^{n} n(n-1) \cdot {}^{n-2}C_{x-2}\, p^{x-2} p^{2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{(n-2)-x-2} + np$$

$$= n(n-1)\, p^2 (p+q)^{n-2} + np$$

$$\boxed{\mu_2' = n(n-1)p^2 + np}$$

If $r = 3$

$$\mu_3' = \sum_{x=0}^{n} x^3 p(x)$$

$$= \sum_{x=0}^{n} \left(x(x-1)(x-2) + 3x(x-1) + x\right) {}^{n}C_{x}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x} + 3\sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x}$$

$$+ \sum_{x=0}^{n} x \cdot {}^nC_x \, p^x \cdot q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2) \frac{n(n-1)(n-2)}{x(x-1)(x-2)} \, {}^{n-3}C_{x-3} \, p^{x-3} p^3 q^{(n-3)-(x-3)}$$

$$+ 3n(n-1)p^2 + np$$

$$= n(n-1)(n-2)p^3 \sum_{x=3}^{n} {}^{n-3}C_{x-3} \, p^{x-3} q^{(n-3)-(x-3)} +$$

$$3n(n-1)p^2 + np$$

$$= n(n-1)(n-2)p^3 (p+q)^{n-3} + 3n(n-1)p^2 + np$$

$$\boxed{\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np}$$

If $r = 4$

$$\mu_4' = \sum_{x=0}^{n} x^4 \, p(x)$$

$$= \sum_{x=0}^{n} [x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x] \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3) \binom{n}{x} p^x q^{n-x} + 6\sum_{x=0}^{n} x(x-1)(x-2)\binom{n}{x} p^x q^{n-x} + 7\sum_{x=0}^{n} x(x-1)\binom{n}{x} p^x q^{n-x}$$

$$+ \sum_{x=0}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} x(x-1)(x-2)(x-3) \frac{n(n-1)(n-2)(n-3)}{x(x-1)(x-2)(x-3)} \, {}^{n-4}C_{x-4}$$

$$p^{x-4} p^4 \cdot q^{(n-4)-(x-4)} + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$= n(n-1)(n-2)(n-3)\,p^4 \sum_{x=4}^{n} {}^{n-4}C_{x-4}\, p^{x-4} q^{(n-4)-(x-4)} + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$= n(n-1)(n-2)(n-3)\,p^4 (p+q)^{n-4} + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

$$\boxed{\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np}$$

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5M Imp

Central moments: (i) $\mu_1 = 0$

V. Imp (ii) $\boxed{\mu_2 = \mu_2' - (\mu_1')^2}$

$$= n(n-1)p^2 + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(1-p) \qquad [\because 1-p = q]$$

$$\boxed{\mu_2 = npq} \text{ - Variance}$$

(iii) $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$$= npq(q-p)$$

(iv) $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$$= npq\{1 + 3(n-2)pq\}$$

Skewness :-

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(npq)^2(q-p)^2}{(npq)^3} = \frac{(q-p)^2}{npq}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq\{1+3(n-2)pq\}}{(npq)^2} = \frac{1+3npq-6pq}{npq}$$

$$\beta_2 = \frac{1-6pq}{npq} + \frac{3npq}{npq}$$

$$\boxed{\beta_2 = \frac{1-6pq}{npq} + 3}$$

Kurtosis:-

$$\gamma_1 = \sqrt{\beta_1} = \sqrt{\frac{(q-p)^2}{npq}} = \frac{(q-p)}{\sqrt{npq}}$$

$$\gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq} + 3 - 3$$

$$= \frac{1-6pq}{npq}$$

Imp

*(MGF) Moment Generating function: The moment Generating function of random variable x is denoted by $M_x(t) = E(e^{tx})$

$$= \sum e^{tx} p(x) \text{ for discrete}$$

$$= \int e^{tx} f(x)\,dx \text{ for continuous}$$

$M_x(t)$ is exists iff the series or integral is absolutely converged.

MGF of Binomial distribution function: we know that MGF is denoted by $M_x(t)$ is and is defined as $M_x(t) = E(e^{tx})$

$$= \sum e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} ({}^nC_x) p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x q^{n-x}$$

$$= {}^nc_0 (pe^t)^0 q^{n-0} + {}^nc_1 (pe^t)^1 q^{n-1} \cdots\cdots +$$
$${}^nc_n (pe^t)^n q^{n-n}$$

$$\boxed{M_x(t) = (q + pe^t)^n}$$

Moment through MGF: // Mean and variance We know that

$$\mu_r' = \left[\frac{d^r}{dt^r} M_x(t)\right]_{t=0}$$

if $r=1$

$$\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0} = \left[\frac{d}{dt} (q+pe^t)^n\right]_{t=0}$$

$$= \left[n (q+pe^t)^{n-1} \cdot pe^t\right]_{t=0}$$

$$\mu_1' = \left[n (q+pe^0)^{n-1} \cdot pe^0\right] \qquad [\because (q+p)=1]$$

$$= \left[n (q+p)^{n-1} \cdot p\right] = np = \text{mean}$$

if $r=2$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt} n(q+pe^t)^{n-1} \cdot pe^t\right]_{t=0}$$

$$= \left[np \frac{d}{dt} (q+pe^t)^{n-1} e^t\right]_{t=0}$$

$$= \left[np \{n-1 (q+pe^t)^{n-2} (pe^t) e^t + (q+pe^t)^{n-1} e^t\}\right]_{t=0}$$

$$= \left[np \{(n-1) (q+pe^t)^{n-2} pe^{2t} + (q+pe^t)^{n-1} e^t\}\right]_{t=0}$$

$$= \left[np \{(n-1) (q+pe^0)^{n-2} pe^{2(0)} + (q+pe^0)^{n-1} e^0\}\right]$$

$= [np\{(n-1)(q+p)^{n-2}p + (q+p)^{n-1}\}]$

$= [np\{(n-1)(q+p)^{n-2}p + (q+p)^{n-1}\}]$

$= np\{(n-1)p+1\}$

$$\boxed{\mu_2' = n(n-1)p^2 + np}$$

If $r=3$

$\mu_3' = \left[\frac{d}{dt} M_x''(t)\right]_{t=0}$

$= \left[\frac{d}{dt}[np\{(n-1)(q+pe^t)^{n-2}pe^{2t} + (q+pe^t)^{n-1}e^t\}]\right]_{t=0}$

$= np\left[(n-1)p\{\frac{d}{dt}(q+pe^t)^{n-2}e^{2t}\} + \frac{d}{dt}(q+pe^t)^{n-1}e^t\right]_{t=0}$

$= np\left[(n-1)p\{(n-2)(q+pe^t)^{n-3}pe^t e^{2t} + (q+pe^t)^{n-2}2e^{2t}\} + (n-1)(q+pe^t)^{n-2}pe^t \cdot e^t + (q+pe^t)^{n-1} \cdot e^t\right]_{t=0}$

$= np\left[(n-1)p\{(n-2)(q+pe^t)^{n-3}pe^{3t} + (q+pe^t)^{n-2}2e^{2t}\} + (n-1)(q+pe^t)^{n-2}pe^{2t} + (q+pe^t)^{n-1}e^t\right]_{t=0}$

$= np[(n-1)p\{(n-2)(q+pe^0)^{n-3}pe^{3(0)} + (q+pe^0)^{n-2}2e^{2(0)}\} + (n-1)(q+pe^0)^{n-2}pe^{2(0)} + (q+pe^0)^{n-1}e^0]$

$= np[(n-1)p\{(n-2)(q+p)^{n-3}p + (q+p)^{n-2}2\} + (n-1)(q+p)^{n-2}p + (q+p)^{n-1}]$

$\mu_3' = np[(n-1)[(n-2)p+2] + (n-1)p + 1]$

$= np[(n-1)(n-2)p^2 + 2(n-1)p + (n-1)p + 1]$

$= n(n-1)(n-2)p^3 + 2n(n-1)p^2 + n(n-1)p^2 + np$

$$\boxed{\mu_3' = n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np}$$

If $r=4$

$$\mu_4' = \left[\frac{d}{dt} M_x'''(t)\right]_{t=0}$$

$$= \left[\frac{d}{dt}\, np\{(n-1)p\{(n-2)(q+pe^t)^{n-3} pe^{3t} + (q+pe^t)^{n-2} 2e^{2t}\} + [(n-1)(q+pe^t)^{n-2} pe^{2t} + (q+pe^t)^{n-1} e^t]\}\right]_{t=0}$$

$$= np\left[\{(n-1)(n-2)p^2 \{\tfrac{d}{dt}(q+pe^t)^{n-3} e^{3t}\} + (2\tfrac{d}{dt}(q+pe^t)^{n-2} e^{2t}) + (n-1)p\tfrac{d}{dt}(q+pe^t)^{n-2} e^{2t}) + \tfrac{d}{dt}(q+pe^t)^{n-1} e^t\}\right]_{t=0}$$

$$= np\{(n-1)p[(n-2)p[\tfrac{d}{dt}(q+pe^t)^{n-3} e^{3t}] + 2\tfrac{d}{dt}(q+pe^t)^{n-2} e^{2t}] + (n-1)p[\tfrac{d}{dt}(q+pe^t)^{n-2} e^{2t}] + \tfrac{d}{dt}(q+pe^t)^{n-1} e^t\}_{t=0}$$

$$= np\{(n-1)p[(n-2)p\{(n-3)(q+pe^t)^{n-3} pe^t e^{3t} + (q+pe^t)^{n-3} 3e^{3t}\} + 2[(n-2)p+2)] + (n-1)p[(n-2)p+2] + (n-1)p+1]\}_{t=0}$$

$$= np\{(n-1)p[(n-2)p\{(n-3)p+3\} + 2(n-2)p+4] + (n-1)(n-2)p^2 + 2(n-1)p + (n-1)p + 1\}$$

$$= np\{(n-1)p[(n-2)(n-3)p^2 + 3(n-2)p + 2(n-2)p + 4] + (n-1)(n-2)p^2 + 2(n-1)p + (n-1)p + 1\}$$

$$= np\{(n-1)(n-2)(n-3)p^3 + 3(n-1)(n-2)p^2 + 2(n-1)(n-2)p^2 + 4(n-1)p + (n-1)(n-2)p^2 + 2(n-1)p + (n-1)p + 1\}$$

$$= n(n-1)(n-2)(n-3)p^4 + 3n(n-1)(n-2)p^3 + 2n(n-1)(n-2)p^3 + 4n(n-1)p^2 + n(n-1)(n-2)p^3 + 2n(n-1)p^2 + n(n-1)p^2 + np$$

$$\boxed{\mu_4' = n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np}$$

09/07/2025 Wednesday

Imp

*characteristic function: The characteristic function of a random variable X is denoted by ϕ_x and is defined as $\phi_x = E(e^{itx}) = \begin{cases} \sum e^{itx} p(x); & \text{for discrete} \\ \int e^{itx} f(x)dx; & \text{for continous} \end{cases}$

characteristic function of Binomial distribution:

we know that characteristic function (C.F) is denoted $\phi_x(t)$ and is defined as

$$\phi_x(t) = E(e^{itx})$$

$$= \sum e^{itx} p(x)$$

$$= \sum e^{itx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^{it})^x q^{n-x}$$

$$= {}^nc_0 (pe^{it})^0 q^{n-0} + {}^nc_1 (pe^{it})^1 q^{n-1} + \dots + {}^nc_n (pe^{it})^n q^{n-n}$$

$$\boxed{\phi_x(t) = (q + pe^{it})^n}$$

*Cummulative Generating function:

The C.G.F of a random variable x is denoted by $K_x(t)$ and is defined as

$$K_x(t) = \log M_x(t)$$

C.G.F of Binomial distribution: we know that C.G.F is denoted by $K_x(t)$ and is defined as $K_x(t) = \log M_x(t)$

$$= \log (q+pe^t)^n$$

$$= n \log (q+pe^t) \qquad [\because q+p=1]$$

$$= n \log \left(q+p\left(1+ t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right)\right)$$

$$= n \log \left(q+p+p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\dots\right)\right)$$

$$\boxed{K_x(t) = n\log\left(1+p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}\dots\right)\right)}$$

$$\left[\because \log(1+x) = \left(x-\frac{x^2}{2!}+\frac{x^3}{3!}-\frac{x^4}{4!}+\dots\right)\right]$$

$$= n\left\{p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}\dots\right) - \frac{p^2}{2!}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}\dots\right)^2 + \frac{p^3}{3!}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}\dots\right)^3 - \frac{p^4}{4!}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots\right)^4\right\}$$

$$K_1 t + K_2\frac{t^2}{2!} + K_3\frac{t^3}{3!} + K_4\frac{t^4}{4!} = n\left\{p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots\right)\right.$$

$$-\frac{p^2}{2!}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\dots\right)^2 + \frac{p^3}{3!}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}\dots\right)^3$$

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$$-\frac{p^4}{4!}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\cdots\right)^4\Big\}$$

comparing coefficients of t, t^2, t^3, t^4 on both sides

$$\boxed{k_1 = n\{p(1)\} = np = \mu_1' = Mean}$$

$$\frac{k_2}{2!} = n\left\{p\left(\frac{1}{2!}\right) - \frac{p^2}{2!}(1)\right\}$$

$$\frac{k_2}{2!} = n\left\{\frac{p}{2} - \frac{p^2}{2}\right\}$$

$$\frac{k_2}{\cancel{2}} = n\left\{\frac{p-p^2}{\cancel{2}}\right\}$$

$$k_2 = np(1-p)$$

$$\boxed{k_2 = npq = Variance}$$

$$\frac{k_3}{3!} = n\left\{p\left(\frac{1}{3!}\right) - \frac{p^2}{2!} + \frac{p^3}{3!}(1)\right\}$$

$$\frac{k_3}{3!} = n\left\{\frac{p}{6} - \frac{p^2}{2} + \frac{p^3}{3}\right\}$$

$$\frac{k_3}{6} = n\left\{\frac{p - 3p^2 + 2p^3}{6}\right\}$$

$$= n\{p - 3p^2 + 2p^3\}$$

$$= np(1-3p+2p^2)$$

$$= np(2p^2 - 2p - p + 1)$$

$$= np(2p(p-1) - 1(p-1))$$

$$= np[(-(1-2p))(-(1-p))]$$

$$= np[(1-p-p)(q)]$$

$$\boxed{K_3 = np[(q-p)(q)]}$$

11/07/2025

Friday

*Recurrence relation for the moments of binomial Distribution: we know that r^{th} moment about mean is denoted by μ_r and is defined as

$$\mu_r = E(X-E(X))^r$$

$$= E(x-\text{mean})^r$$

$$= E(x-np)^r$$

$$= \sum_{x=0}^{n} (x-np)^r p(x)$$

diff w.r.to 'p'

$$\frac{d}{dp}\mu_r = \frac{d}{dp}\left[\sum_{x=0}^{n} (x-np)^r p(x)\right]$$

$$= \sum_{x=0}^{n}\left[\frac{d}{dp}(x-np)^r p(x)\right]$$

$$= \sum_{x=0}^{n}\left[r(x-np)^{r-1}(-n).p(x)+(x-np)^r p'(x)\right] \quad (1)$$

w.k.t $p(x) = \binom{n}{x} p^x q^{(n-x)}$

$$p'(x) = {}^nC_x\left[x.p^{x-1}q^{n-x}+p^x(n-x)q^{(n-x-1)}(-1)\right]$$

$$= {}^nC_x\left[x p^{x-1} q^{n-x} - p^x(n-x)q^{n-x}q^{-1}\right]$$

$$= {}^nC_x\, p^x q^{n-x}\left[xp^{-1}-(n-x)q^{-1}\right]$$

$$= p(x)\left[\frac{x}{p} - \frac{(n-x)}{q}\right]$$

$$= p(x)\left[\frac{qx-np+px}{\ }\right]$$

$$= p(x)\left[\frac{x(q+p) - np}{pq}\right]$$

$$p'(x) = p(x)\left[\frac{x - np}{pq}\right]$$

Sub $p'(x)$ in eq ①

$$\frac{d}{dp}\mu_r = \sum_{x=0}^{n}\left[r \cdot (x-np)^{r-1} p(x)(-n) + (x-np)^r p(x)\left(\frac{x-np}{pq}\right)\right]$$

$$= \sum_{x=0}^{n}\left[-nr(x-np)^{r-1} p(x) + \frac{(x-np)^{r+1}}{pq} \cdot p(x)\right]$$

$$= -nr\sum_{x=0}^{n}(x-np)^{r-1} p(x) + \frac{1}{pq}\sum_{x=0}^{n}(x-np)^{r+1} p(x)$$

$$\frac{d}{dp}\mu_r = -nr\mu_{r-1} + \frac{1}{pq}\mu_{r+1}$$

$$\frac{d}{dp}\mu_r + nr\mu_{r-1} = \frac{1}{pq}\mu_{r+1}$$

$$\mu_{r+1} = pq\left[\frac{d}{dp}\mu_r + nr\mu_{r-1}\right]$$

If $r=0$

$$\mu_{0+1} = pq\left[\frac{d}{dp}\mu_0 + (0)\, n\, \mu_{0-1}\right]$$

$$\mu_1 = pq\left[\frac{d}{dp} 1 + 0\right] \qquad [\mu_0 = 1]$$

$$\mu_1 = 0$$

If $r=1$

$$\mu_{1+1} = pq\left[\frac{d}{dp}\mu_1 + 1\, n\, \mu_{1-1}\right]$$

$$= pq\left[\frac{d}{dp}0 + n\mu_0\right]$$ $[\because \mu_1 = 0]$

$$= pq\,n(1)$$

$$\mu_2 = npq$$

If $r = 2$

$$\mu_{2+1} = pq\left[\frac{d}{dp}\mu_2 + n(2)\mu_{2-1}\right]$$

$$= pq\left[\frac{d}{dp}\mu_1 npq + 2n\mu_1\right]$$

$$= pq\left[\frac{d}{dp}npq + 2n(0)\right]$$

$$= npq\left[\frac{d}{dp}p(1-p)\right]$$

$$= npq\left[\frac{d}{dp}p - p^2\right]$$

$$= npq\left[1 - 2p\right]$$

$$= npq\left[1 - p - p\right]$$

$$\mu_3 = npq\left[q - p\right]$$

14/07/2025 Monday

If $r = 3$

$$\mu_{3+1} = pq\left[\frac{d}{dp}\mu_3 + 3n\mu_{3-1}\right]$$

$$= pq\left[\frac{d}{dp}npq(q-p) + 3n\mu_2\right]$$

$$= pq\left[\frac{d}{dp}npq(q-p) + 3n(npq)\right]$$

$$= npq\left[\frac{d}{dp}pq(q-p) + 3npq\right]$$

$$= npq\left[\frac{d}{dp}\, p(1-p)(q-p) + 3npq\right] \qquad [\because q = 1-p]$$

$$= npq\left[\frac{d}{dp}\, p(1-p)(1-p-p) + 3np(1-p)\right]$$

$$= npq\left[\frac{d}{dp}(p-p^2)(1-p-p) + 3np(1-p)\right]$$

$$= npq\left[\frac{d}{dp}(p-p^2)(1-2p) + 3npq\right]$$

$$= npq\left[\frac{d}{dp}(p-2p^2-p^2+2p^3) + 3npq\right]$$

$$= npq\left[\frac{d}{dp}(1-4p-2p+6p^2) + 3npq\right]$$

$$= npq\left[(1-6p+6p^2) + 3npq\right]$$

$$= npq\left[1-6p(1-p) + 3npq\right]$$

$$= npq\left[1-6pq + 3npq\right]$$

$$\boxed{\mu_4 = npq\left[1+3pq(n-2)\right]}$$

* Recurrence relation for the probabilities of Binomial distribution: we know that $p(x) = \binom{n}{x} p^x q^{n-x}$

$$p(x+1) = {}^nC_{x+1}\, p^{x+1} q^{n-(x+1)}$$

Consider $\dfrac{p(x+1)}{p(x)} = \dfrac{{}^nC_{x+1}\, p^{x+1} q^{n-(x+1)}}{{}^nC_x\, p^x q^{n-x}}$ $\left[\because {}^nC_x = \dfrac{n!}{(n-x)!\,x!}\right]$

$= (n-x-1)!$

$$\frac{p(x+1)}{p(x)} = \frac{\dfrac{n!}{(n-x-1)!\,(x+1)!}\; p^x\cdot p\, q^{n-x}\, q^{-1}}{\dfrac{n!}{(n-x)!\,x!}\; p^x\cdot q^{n-x}}$$

$(n-x+1+1)$

$(n-x)$

$= x!$

$= (x+1)x!$

$$\frac{p(x+1)}{p(x)} = \frac{p}{q}\left[\frac{(n-x)(n-x-1)!\,(x)!}{(n-x-1)!\,(x+1)\,x!}\right]$$

$$\frac{p(x+1)}{p(x)} = \frac{p}{q}\left[\frac{n-x}{(x+1)}\right]$$

$$\boxed{p(x+1) = p(x)\left[\frac{n-x}{x+1}\cdot\frac{p}{q}\right]}$$

//

Imp

* probability Generating function of Binomial distribution

we know that the p.G.f is denoted by $P_X(s)$ and is defined as $P_X(s) = E[s^x]$

$$= \sum_{x=0}^{n} s^x p(x)$$

$$= \sum_{x=0}^{n} s^x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (sp)^x q^{n-x}$$

$$= {}^nC_0 (ps)^0 q^{n-0} + {}^nC_1 (ps)^1 q^{n-1} + \ldots\ldots + {}^nC_n (ps)^n q^{n-n}$$

$$\boxed{P_X(s) = (q+ps)^n}$$ //

* Additive property of Binomial distribution :-

let X and Y be two independent binomial variables with the parameters $n_1 p_1$ and $n_2 p_2$.

we know that $M_X(t) = (q_1 + p_1 e^t)^{n_1}$

$$M_Y(t) = (q_2 + p_2 e^t)^{n_2}$$

The distribution of $X+Y$ is

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$$

$$\boxed{M_{X+Y}(t) = (q_1+p_1e^t)^{n_1} \cdot (q_2+p_2e^t)^{n_2}}$$

It cannot be expressed in the form of M.G.F of Binomial distribution, hence the distribution X+Y is not a binomial, therefore the binomial distribution does not follow additive property.

But if $p_1 = p_2 = p$, $q_1 = q_2 = q$

$$M_{(X+Y)}t = (q+pe^t)^{n_1}(q+pe^t)^{n_2}$$

$$\boxed{M_{(X+Y)}t = (q+pe^t)^{n_1+n_2}}$$

Hence, By uniqueness theorem of M.G.F the distribution of X+Y is binomial with the parameters ~~$(n_1+n_2)p$~~ (n_1+n_2, p).

problem-1: The M.G.F of random variable X is $\left(\frac{2}{3}+\frac{1}{3}e^t\right)^9$ Find the distribution.

Sol: M.G.F of binomial distribution is $(q+pe^t)^n$

$$= \left(\frac{2}{3}+\frac{1}{3}e^t\right)^9 \quad q = \frac{2}{3}, \; p = \frac{1}{3}, \; n = 9.$$

$$p(x) = \binom{n}{x} p^x q^{n-x}$$

$$p(x) = \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

problem-2: 10 Coins are tossed simultaneously. Find the probability of getting atleast 7 heads.

Sol $n=10,\ p=1/2,\ q=1/2$

$$p(x\geq 7) = p(x=7)+p(x=8)+p(x=9)+p(x=10)$$

$$= {}^{10}C_7 (1/2)^7 (1/2)^{10-7} + {}^{10}C_8 (1/2)^8 (1/2)^{10-8} + {}^{10}C_9 (1/2)^9 \left(\tfrac{1}{2}\right)^{10-9}$$

$$+ {}^{10}C_{10} (1/2)^{10} (1/2)^{10-10}$$

$$= {}^{10}C_7 (1/2)^7 (1/2)^3 + {}^{10}C_8 (1/2)^8 (1/2)^2 + {}^{10}C_9 (1/2)^9 (1/2)^1$$

$$+ {}^{10}C_{10} (1/2)^{10}$$

$$= \left(\tfrac{1}{2}\right)^{10} \left[{}^{10}C_7 + {}^{10}C_8 + {}^{10}C_9 + {}^{10}C_{10}\right]$$

$$= \frac{1}{1024}\left[120 + 48 + 10 + 1\right] = 0.174$$

S.M. Shaansi

* Limiting case of binomial distribution to normal distribution:

Binomial distribution tends to normal distribution as $n \to \infty$ (or)

m.g.f of standard Binomial variate $\frac{X-np}{\sqrt{npq}}$ tends to $e^{t^2/2}$ as $n \to \infty$

proof: If $X \sim B(n,p)$, then m.g.f is

$$M_X(t) = (q+pe^t)^n$$

let the standard binomial variate $Z = \frac{X-np}{\sqrt{npq}}$,

then m.g.f of Z is

$$M_Z(t) = E(e^{tZ})$$

$$= E\left(e^{t\left(\frac{X-np}{\sqrt{npq}}\right)}\right)$$

$$= e^{-\frac{tnp}{\sqrt{npq}}} \cdot E\left(e^{\frac{tX}{\sqrt{npq}}}\right)$$

$$= e^{-\frac{tnp}{\sqrt{npq}}} \cdot M_x\left(t/\sqrt{npq}\right)$$

Binomial

$$= e^{-\frac{tnp}{\sqrt{npq}}} \cdot \left(q + pe^{\frac{t}{\sqrt{npq}}}\right)^n$$

$$= \left[e^{-\frac{tp}{\sqrt{npq}}} \cdot \left(q + pe^{\frac{t}{\sqrt{npq}}}\right)\right]^n$$

$$= \left(qe^{-\frac{tp}{\sqrt{npq}}} + pe^{\frac{tq}{\sqrt{npq}}}\right)^n$$

$$= \left[q\left(1 - \frac{tp}{\sqrt{npq}} + \frac{t^2p^2}{2!\,npq} - + \cdots\right) + p\left(1 + \frac{tq}{\sqrt{npq}} + \frac{t^2q^2}{2!\,npq} + \cdots\right)\right]^n$$

$$= \left[q + p + \frac{t^2}{2} \cdot \frac{pq(p+q)}{npq} + o(n^{-3/2})\right]^n$$

$$= \left[1 + \frac{t^2}{2n} + o(n^{-3/2})\right]^n$$

where $o(n^{-3/2})$ is the terms containing powers $\frac{3}{2}$ and more of n in the denominator

consider logarithms,

$$\log M_z(t) = n\log\left[1 + \frac{t^2}{2n} + o(n^{-3/2})\right]$$

$$= n\left[\frac{t^2}{2n} + o(n^{-3/2}) - \frac{1}{2}\left(\frac{t^2}{2n} + o(n^{-3/2})\right)^2 + \cdots\right]$$

$$= \frac{t^2}{2} + o(n^{-1/2})$$

where $o(n^{-1/2})$ is the terms containing power $\frac{1}{2}$ and more of n in the denominator.

Apply limits as $n \to \infty$

$$\lim_{n \to \infty} \log M_Z(t) = \frac{t^2}{2}$$

Consider exponential

$$\lim_{n \to \infty} M_Z(t) = e^{t^2/2}$$

which is the m.g.f of standard normal distribution. Hence by uniqueness theorem of m.g.f, standard Binomial normal variate tends to standard normal variate. Therefore binomial distribution tends to normal distribution, as $n \to \infty$.

I th...